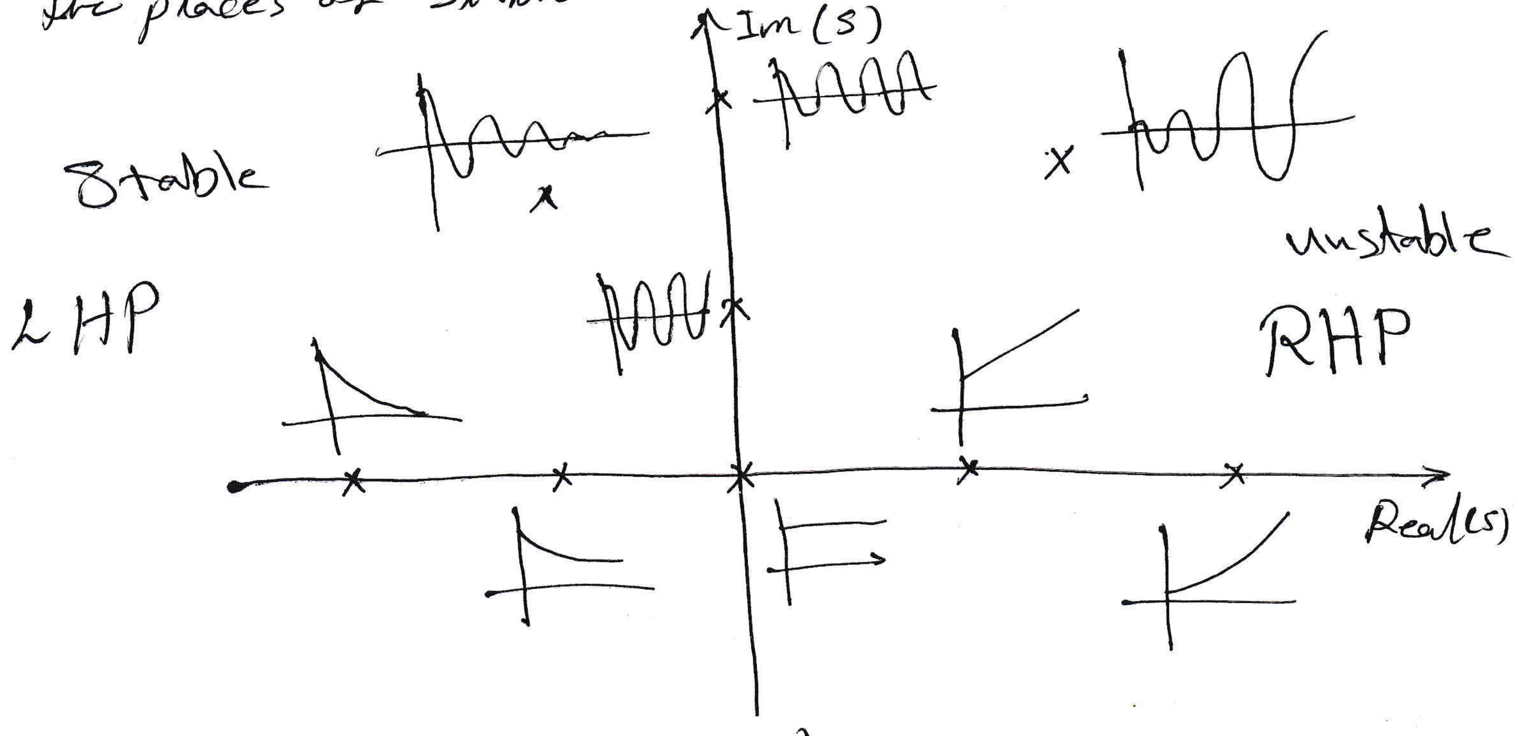
# 6 - Stability

A linear time-invariant system is said to be stable if all the roots of the transfer function denominator poly have negative real parts (i.e., they are all in the left hand s-plane) and is unstable otherwise.

A system is stable if its initial conditions decay to zero and is unstable if they diverge. The system is "stable" if all the poles of the system are inside the left half s-plane [i.e., all the poles have negative real parts ($\sigma < 0$)]. If "any" pole of the system is in the right half s-plane (i.e., has a positive real part $\sigma > 0$), then the system is "unstable". If $\sigma = 0$, then the oscillatory motion will persist. Figure below shows the places of stable and unstable on s-plane.



Assum the transfer function, has this formula

$$T(s) = \frac{Y(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n} \quad \text{---} (1)$$

$$= \frac{K \prod_{i=1}^{m} (s - z_i)}{\prod_{i=1}^{n} (s - p_i)}, \quad m \leq n$$

The denominator of the transfer function is called " the characteristic equation". the roots [$P_i$] of the characteristic equation are real or Complex, but are "distinct".

The solution to the differental equation whose characteristic equation is given by

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_0 = 0 \quad \text{----} (2)$$

may be written using partial fraction expression as

$$y(t) = \sum_{i=1}^{n} K_i e^{p_i t}$$

$P_i$ : are the roots of eq (2)

$K_i$ : depends on the initial conditions and zero locations.

for example

$$D(s) = s^3 + s^2 + 3s + 24 = 0$$

$$= (s - 1 + j2\sqrt{2})(s - 1 - j2\sqrt{2})(s + 3) = 0$$

there are two complex numbers in RHP and one in the left LHP, that means the system is unstable.

## 6.1 Routh Criterion Stability :-

This Criteria determines how many roots of the characteristic equation lie in the right half of s-plane. This test also determines all the roots on the $j\omega$-axis, so that their multiplicity can be found out. The polynomial coefficients are arranged in an array called Routh array.

Let the characteristic equation be given by,

$$D(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_1 s + a_0 = 0$$

$$\begin{array}{c|cccc}
s^n & a_0 & a_2 & a_4 & \cdots\ a_n \\
s^{n-1} & a_1 & a_3 & a_5 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & \\
s^{n-3} & c_1 & c_2 & c_3 & \\
\vdots & & & &
\end{array}$$

the coefficients in $s^{n-2}$ row are obtained as follows:

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$ and so on

the coefficients of $s^{n-3}$ row are obtained in a similar way, considering the coefficients of the previous two rows as follows:-

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$ and so on

EX Consider the characteristic equation,

$$D(s) = s^4 + 2s^3 + 8s^2 + 4s + 3 = 0$$

Comment on its stability.

| | | | |
|---|---|---|---|
| $s^4$ | 1 | 8 | 3 |
| $s^3$ | 2 | 4 | 0 |
| $s^2$ | $\frac{2\times8-1\times4}{2}=6$ | $\frac{2\times3-1\times0}{2}=3$ | |
| $s^1$ | $\frac{6\times4-2\times3}{6}=3$ | | |
| $s^0$ | 3 | | |

In the first column 1, 2, 6, 3, 3, All are positive and therefore, the system is "stable".

EX Examine the characteristic equation

$$D(s) = s^4 + 2s^3 + s^2 + 4s + 2 = 0 \quad \text{for stability}$$

| | | | |
|---|---|---|---|
| $s^4$ | 1 | 1 | 2 |
| $s^3$ | 2 | 4 | |
| $s^2$ | $\frac{2-4}{2} = -1$ | 2 | |
| $s^1$ | $\frac{-4-4}{-1} = 8$ | | |
| $s^0$ | 2 | | |

the first column, there are 1, 2, -1, 8 and 2. one of the coefficients is negative and hence the system is unstable. Also, there are two sign changes, 2 to -1 and -1 to +8. Hence there are two two roots of the characteristic equation in the right of s-plane.

## 6-2 Special Cases

(i) First case:

If there is "Zero" in Routh arry. we can solve this problem by two way.

(a) First method:

Replacing the Zero in that row by "$\epsilon$". Proceed with the construction of the table.

EX Consider the characteristic equation

$$D(s) = s^5 + s^4 + 3s^3 + 3s^2 + 6s + 4$$

Comment on the stability.

$$\begin{array}{l|ccc} s^5 & 1 & 3 & 6 \\ s^4 & 1 & 3 & 4 \\ s^3 & \frac{1*3-1*3}{1} = 0 & \frac{6-4}{1} = 2 & \end{array}$$

$$\begin{array}{lccc} s^5 & 1 & 3 & 6 \\ s^4 & 1 & 3 & 4 \\ s^3 & \epsilon & 2 & \\ s^2 & \frac{3\epsilon-2}{\epsilon} & 4 & \\ s^1 & \frac{\frac{6\epsilon-4}{\epsilon} - 4\epsilon}{\frac{3\epsilon-2}{\epsilon}} = \frac{6\epsilon-4-4\epsilon^2}{3\epsilon-2} & & \\ s^0 & 4 & & \end{array}$$

$s^3$ $\lim_{\epsilon \to 0} \epsilon = 0$

$s^2$ $\lim_{\epsilon \to 0} \frac{3\epsilon-2}{\epsilon} = \infty$

$s^1 = \lim_{\epsilon \to 0} \frac{6\epsilon-4-4\epsilon^2}{3\epsilon-2} = 2$

$s^0$ 4

Hence the elements of $1^{st}$ column of Routh array are $1, 1, 0, -\infty, 2$, and $4$. there are two sign changes and hence there are two roots in the right half of s-plane. the system therefore is unstable.

b) Second method :-

by replacing $s$ by $\frac{1}{Z}$ in $D(s)$ and apply the Routh Criterion for the resulting equation in $Z$.

ex Comment the stability of the characteristic equation

$$D(s) = s^5 + s^4 + 3s^3 + 3s^2 + 6s + 4$$

$$\frac{1}{Z^5} + \frac{1}{Z^4} + \frac{3}{Z^3} + \frac{3}{Z^2} + \frac{6}{Z} + 4 = 0$$

$$4Z^5 + 6Z^4 + 3Z^3 + 3Z^2 + Z + 1 = 0$$

| | | | |
|---|---|---|---|
| $Z^5$ | $4$ | $3$ | $1$ |
| $Z^4$ | $6$ | $3$ | $1$ |
| $Z^3$ | $\frac{18-12}{6} = 1$ | $\frac{1}{3}$ | |
| $Z^2$ | $\frac{3-2}{1} = 1$ | $1$ | |
| $Z^1$ | $\frac{\frac{1}{3}-1}{1} = -2$ | | |
| $Z^0$ | $1$ | | |

ii) Second Case:

For some systems, a particular row may contain all zero entries. This happens when the characteristic equation contains roots which are symmetrically located about real and imaginary axes, namely:

a) one or more pairs of roots on the $j\omega$-axis

b) one or more pairs of real roots with opposite signs, and

EX Consider

$$D(s) = s^6 + s^5 + 6s^4 + 5s^3 + 10s^2 + 5s + 5$$

obtain the number of roots in the RHS of s-plane.

| | | | | |
|---|---|---|---|---|
| $s^6$ | 1 | 6 | 10 | 5 |
| $s^5$ | 1 | 5 | 5 | |
| $s^4$ | $\frac{6-5}{1} = 1$ | 5 | 5 | |
| $s^3$ | 0 | 0 | | |

there are two zeros in $s^3$ row. The auxiliary polynomial coefficient is

$$A(s) = s^4 + 5s^2 + 5$$

$$\frac{dA(s)}{ds} = 4s^3 + 10s$$

Now, replacing the $s^3$ row in the Routh table with the coefficients of $\frac{dA(s)}{ds}$, we have

| | | | | |
|---|---|---|---|---|
| $s^6$ | 1 | 6 | 10 | 5 |
| $s^5$ | 1 | 5 | 5 | |
| $s^4$ | 1 | 5 | 5 | |
| $s^3$ | 4 | 10 | | |
| $s^2$ | $\frac{20-10}{4} = 2.5$ | 5 | | |
| $s^1$ | 2 | | | |
| $s^0$ | 5 | | | |

We see, there are no sign changes. But we have symmetrically located roots, we have to find these roots for the stability. Factoring the auxiliary polynomial, we have the root as,

$$\pm j1.1756 \text{ and } \pm j1.902$$

the system is limited stable

EX Comment on the stability of the system with the following characteristic equation.

$$D(s) = s^6 + s^5 + 7s^4 + 6s^3 + 31s^2 + 25s + 25$$

| | | | | |
|---|---|---|---|---|
| $s^6$ | 1 | 7 | 31 | 25 |
| $s^5$ | 1 | 6 | 25 | |
| $s^4$ | 1 | 6 | 25 | |
| $s^3$ | 0 | 0 | | |
| $s^2$ | | | | |
| $s^1$ | | | | |
| $s^0$ | | | | |

Since there is a row of zeros, let us construct the auxiliary polynomial.

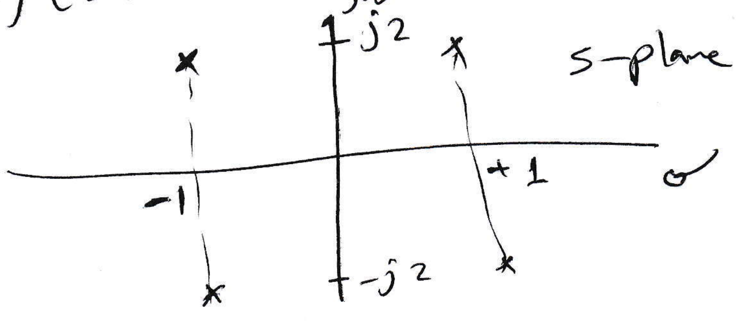
$$A(s) = s^4 + 6s^2 + 25$$

$$\frac{dA(s)}{ds} = 4s^3 + 12s$$

| | | | | |
|---|---|---|---|---|
| $s^6$ | 1 | 7 | 31 | 25 |
| $s^5$ | 1 | 6 | 25 | |
| $s^4$ | 1 | 6 | 25 | |
| $s^3$ | 4 | 12 | | |
| $s^2$ | 3 | 25 | | |
| $s^1$ | $-\frac{64}{3}$ | | | |
| $s^0$ | 25 | | | |

There are two sign changes in the first column of the Routh table and hence the system is unstable.

$$
\begin{aligned}
A(s) &= s^4 + 6s^2 + 25 \\
&= (s^2+5)^2 + 6s^2 - 10s^2 \\
&= (s^2+5)^2 - 4s^2 \\
&= \left[(s^2+5) - 2s\right]\left((s^2+5) + 2s\right) \\
&= (s^2 - 2s + 5)(s^2 + 2s + 5) \\
&= (s+1+j2)(s+1-j2)(s-1+j2)(s-1-j2)
\end{aligned}
$$



## 6-3 Routh Hurwitz criterion :

If after completing the Routh table, there are no sign changes, the auxiliary polynomial will have roots only on the jw-axis. they can be found out by factoring the auxiliary polynomial. The procedure to determine stability of a system and find the number of roots in the right half of S-plane, using Routh Hurwitz criterion.

<u>step 1</u> :- The characteristic equation is examined for necessary condition.

(i) All the coefficients must be positive.

(ii) No coefficient of $D(z)$ is zero between the highest and lowest powers of S. If these conditions are satisfied, go to step 2.

<u>step 2</u> :- by using Routh table, if there are "m" sign changes in the column, there are m roots in the right half of S-plane, the system is unstable. If it is required to find the number of roots in RHS of S-plane go to step 3. If all the entries in a row are zero, go to step 4.

**Step 3** :- Using the same procedure before for finding $\epsilon$ Replacing the Zero.

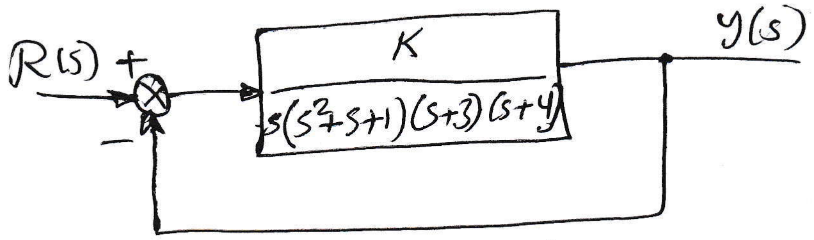
**Step 4** :- Replacing the rows of Zeros with the coefficients of the differential of the polynomial ($A(s)$). Complete the table now.

**Step 5** :- the number of roots in the RHS depends on "m" sign changes in the first column of the table. If there are no sign changes go to step 6.

**Step 6** :- Factorise the polynomial $A(s)$. Since the are no roots in the RHS of s-plane. $A(s)$ will contain roots on the $j\omega$-axis only. Find these roots. If these roots are simple, the system is limited stable. If any of these roots is repeated, the system is unstable. This concludes the procedure.

To find the stability of the system, we must find the feedback loop. It is therefore desirable to know the range of the values of the gain $K$ for maintiaining the system in stable condition.

EX Find the range of values of $K$ for the closed loop System in figure below to remain stable. Find the frequency of sustained oscillations under limiting conditions.




The closed loop transfer function is given by

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$$G(s) = \frac{K}{s(s^2+s+1)(s+3)(s+4)} \quad ; \quad H(s) = 1$$

$$T(s) = \frac{N(s)}{D(s)} = \frac{K}{s(s^2+s+1)(s+3)(s+4) + K}$$

the characteristic equation is given by

$$D(s) = s(s^2+s+1)(s+3)(s+4) + K$$

$$= s^5 + 8s^4 + 20s^3 + 19s^2 + 12s + K$$

| | | | |
|---|---|---|---|
| $s^5$ | 1 | 20 | 12 |
| $s^4$ | 8 | 19 | $K$ |
| $s^3$ | 17.625 | $\frac{96-K}{8}$ | |
| $s^2$ | $\frac{238.875+K}{17.625}$ | $K$ | |

$$
\begin{array}{c|c}
s^1 & \dfrac{\frac{238.875+K}{17.625} * \frac{96-K}{8} - 17.625}{\frac{238.875+K}{17.625}} \\
s^0 & K
\end{array}
$$

~~Examinat~~

Examining the first colum the system will be stable if,

i) $\dfrac{238.875+K}{17.625} > 0$

ii) $\dfrac{(238.875+K)(96-K)}{141} - 17.625\,K > 0$

iii) $K > 0$

if condition (iii) is satisfied condition (i) is automatically satisfied. let us find out the value of $K$, condition (ii) will be satisfied.

$$-K^2 - 142.875K + 22932 - 2485.125K > 0$$

$$K^2 + 2628K - 22932 < 0$$

$$(K - 8.697)(K + 2636.7) < 0$$

since $K > 0$, the above condition is satisfied for $K > 8.697$. thus the range of values of $K$ for stability is $0 < K < 8.697$.

if $K > 8.697$, then the roots of $s^1$ will be negative in R.H.S of s-plane.

if $K < 8.697$, the roots of $s^1$ will be positive and the system will be stable.

if $K = 8.697$, the roots will be zero, it indicates roots on the imaginary axis. $s^2$ row will give these roots.

$$A(s) = \left(\frac{238.875 + 8.697}{17.625}\right) s^2 + 8.697$$

$$= 14.047 s^2 + 8.697$$

the roots of this polynomial are $\pm j0.7869$. if $K = 8.697$, the closed loop system will have a pair of roots at $\pm j0.7869$ and the response will exhibit substained oscillations with a frequency of 0.7869 rad/sec.

EX Examine the stability of the characteristic polynomial for K ranging from 0 to $\infty$.

$$D(s) = s^4 + 20Ks^3 + 5s^2 + 10s + 15$$

| | | | |
|---|---|---|---|
| $s^4$ | $1$ | $5$ | $15$ |
| $s^3$ | $20K$ | $10$ | |
| $s^2$ | $\frac{100K-10}{20K}$ | $15$ | |
| $s^1$ | $\frac{\frac{100K-10}{2K} - 300K}{\frac{100K-10}{20K}}$ | | |
| $s^0$ | $15$ | | |

The system will be stable if

i) $K > 0$

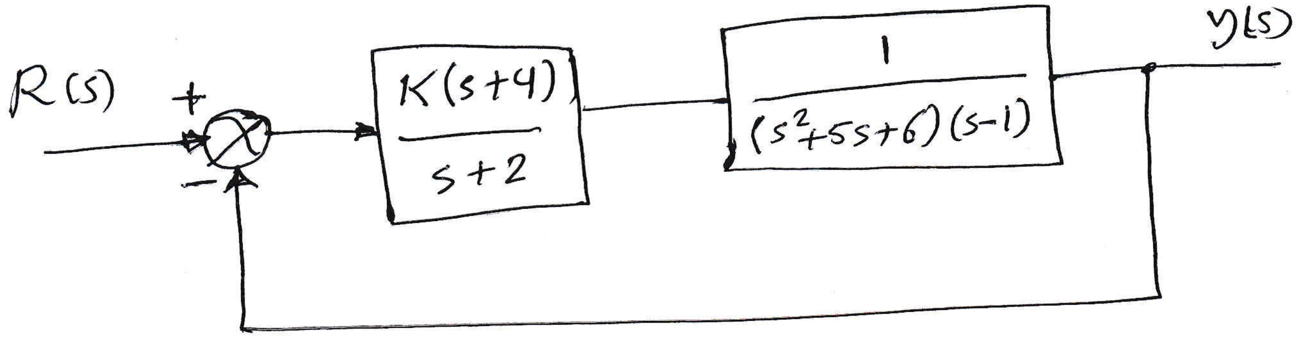
ii) $\frac{100K-10}{20K} > 0$ or $K > 0.1$

iii) $\left(\frac{100K - 10 - 600K^2}{10K-1}\right) > 0$

or $60K^2 - 10K + 1 < 0$

This is not satisfied for any real value of K. Hence for no value of K, the system is stable.

EX Comment on the stability of the closed loop system, as the gain K is changed in figure below.




We notice that G(s) in open loop is unstable because $s = 1$ in the RHS of s-plane. let check the stability with ~~chose~~ varying the amplifier gain K.

the characteristic equation is

$$D(s) = 1 + G(s)H(s) = 0$$

$$1 + \frac{K(s+4)}{(s+2)(s^2+5s+6)(s-1)} = 0$$

$$s^4 + 76s^3 + 11s^2 + s(K-2) + 4(K-4) = 0$$

$$\begin{array}{c|ccc}
s^4 & 1 & 11 & 4(K-2) \\
s^3 & 6 & (K-2) & \\
s^2 & \frac{68-K}{6} & 4(K-2) & \\
s^1 & \dfrac{\left(\frac{68K-10}{6}\right)(K-2) - 24(K-4)}{\frac{68-K}{6}} & & \\
s^0 & 4(K-4) & &
\end{array}$$

for stability

i) $68 - K > 0$, or $K < 68$

ii) $(68-K)(K-2) - 144(K-4) > 0$

iii) $4(K-4) > 0$, or $K > 4$

from condition (ii)

$$-K^2 + 70K - 136 - 144K + 576 > 0$$

or $K^2 + 74K - 440 < 0$

$$K = \frac{-74 \mp \sqrt{74^2 + 1760}}{2}$$

$$= 5.532, -79.532$$

$\therefore (K - 5.532)(K + 79.532) <$

i.e. $-79.532 < K < 5.532$

Combining the conditions (i), (ii) and (iii) we have, for stability $\quad 4 < K < 5.532$

if $K = 4$, the characteristic equation is zero and there will be a pole at the origin. For $K = 5.532$, the elements in the $s^1$ row will all be zeros.

The roots of the auxiliary polynomial are

$$A(s) = \left(\frac{68-K}{6}\right)s^2 + 4(K-4)$$

$K = 5.532$

$A(s) = 10.411\,s^2 + 6.128$

$S = \pm j\,0.7672.$

The frequency of oscillation of response for $K = 5.532$ is $0.7676$ rad/sec.

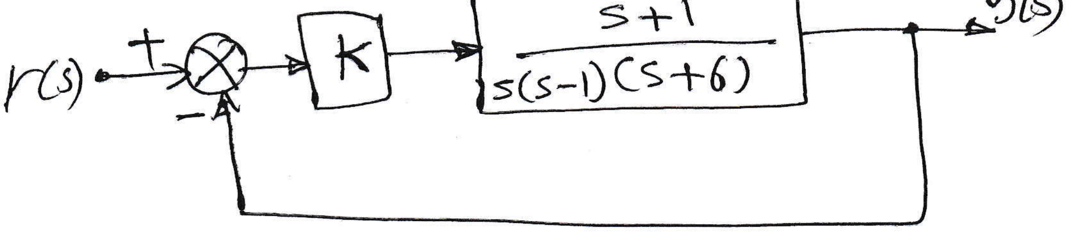
# Stability

# Homework 4

**Q1** Determine wheather of the roots of the polynomial are in the RHP.

$$D(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4$$

**Q2** Determine the range of K over which the system is stable.




**Q3** Find the range of gains (K, $K_I$) for stability

$$D(s) = s^3 + 3s^2 + (2+K)s + K_I = 0$$

**Q4** Consider the polynomial

$$D(s) = s^5 + 3s^4 + 2s^3 + 6s^2 + 6s + 9$$

Determine wheather any of the roots are in the RHP.

**Q5** For the polynomial

$$D(s) = s^5 + 5s^4 + 11s^3 + 23s^2 + 28s + 12.$$

determine wheather there are any roots on the $j\omega$ axis or in the RHP.

Q6 by using Routh stability criterion, determine wheather the resulting closed-loop system will be stable.

a) $$KG(s) = \frac{4(s+2)}{s(s^3+2s^2+3s+4)}$$

b) $$KG(s) = \frac{4(s^3+2s^2+s+1)}{s^2(s^3+2s^2-s-1)}.$$